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Competition between the catalyzed birth and death in the exchange-driven growth

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We propose a three-species (A, B, and C) exchange-driven aggregate growth model with competition between catalyzed birth and catalyzed death. In the system, exchange-driven aggregation occurs between any two aggregates of the same species with the size-dependent rate kernel $K_n(k,j)=K_n kj$ $(n=1,2,3)$, and, meanwhile, monomer birth and death of species A occur under the catalysis of species B and C with the catalyzed birth and catalyzed death rate kernels $I(k,j)=Ikj^{v}$ and $J(k,j)=Jkj^{v}$, respectively. The kinetic behavior is investigated by means of the mean-field rate equation approach. The form of the aggregate size distribution $a_k(t)$ of species A is found to depend crucially on the competition between species-B-catalyzed birth of species A and species-C-catalyzed death of species A, as well as the exchange-driven growth. The results show that (i) when exchange-driven aggregation dominates the process, $a_k(t)$ satisfies the conventional scaling form; (ii) when catalyzed birth dominates the process, $a_k(t)$ takes the conventional or generalized scaling form; and (iii) when catalyzed death dominates the process, the aggregate size distribution of species A evolves only according to some modified scaling forms.

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I. INTRODUCTION

The phenomenon of aggregate growth is popularly studied and important in a wide variety of fields in nature, such as physics, chemistry, and biology. For example, aggregation underlies the evolution of planetary systems in astrophysics [1], cloud formation and dust accumulation in atmospheric sciences [2–4], as well as polymer and gel formation in chemical physics [5–9]. The originally studied mechanism is the pure cluster-cluster aggregation process [10–13], and research was then steadily extended to aggregation phenomena with much more complex mechanisms, such as fragmentation and annihilation, and their various combinations [14–17]. Aggregation also plays a central role in the theory of percolation [18], fractal formation [19], and network growth [20–22].

In recent years, much attention has been devoted to generalized aggregation phenomena in sociology and economy to investigate the kinetic behavior of aggregation growth driven by migration or exchange. Ispolatov, Krapivsky, and Redner introduced several asset exchange models for the evolution of wealth distribution in an economically interaction population [23]. Leyvraz and Redner proposed a migration-driven irreversible aggregate growth model for the evolution of city populations [24]. In these models, irreversible growth of aggregates takes place through biased migration or unbiased exchange mechanisms. The biased migration mechanism can be described by an irreversible reaction scheme $A_k+A_l \xrightarrow{K(k,l)} A_{k-1}+A_{l+1}$ $(k\leqslant l)$, where A_k denotes an aggregate characterized only by its size k (an aggregate containing k units; for example, an aggregate of k units of assets in asset exchange models or k persons in migration-driven city population models). The migration rate kernel $K(k,l)$ represents the rate of monomer migration from an aggregate of size k to another aggregate of size l, which generally depends on the sizes of the two aggregates. In an unbiased exchange model, an aggregate is equally likely to gain or to lose a monomer. Ke and Lin investigated the kinetics of a general unbiased migration-driven aggregation system [25], and Lin and Ke further generalized the research to exchange-driven aggregation with birth and death to mimic the evolutions of city populations and individual wealth [26]. Ben-Naim and Krapivsky made a general study of exchange-driven growth with a product and a generalized homogeneous rate kernel [27]. These migration or exchange-driven aggregation processes exhibit much more abundant kinetic behaviors than those in the single-aggregation, annihilation, fragmentation processes or their various combinations. Ke and Lin and co-workers also generalized the study to the kinetics of migration-driven aggregate growth on completely connected scale-free networks [28]. The results show that the evolution behavior of the aggregate size distribution is drastically different from that for the corresponding system in normal space.

In an aggregate growth system of more than one species, there may exist interactions between two aggregates of different species. Recently, considerable interest has been aroused in a variety of catalysis-driven aggregation mechanisms [29,30]. Lin and Ke and co-workers studied the mutually catalyzed birth of population and assets in exchange-driven growth to investigate the interaction between the population and the asset aggregates [31]. The competition between the two growth mechanisms, exchange and catalyzed birth, gives rise to much more abundant kinetic behavior. Here it is noticed that catalyzed death may play an important role, as well as the catalyzed birth process in the evolutions of city population, wealth distribution, and other natural and social aggregation growths. The study of the competition between the catalyzed birth and catalyzed death

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processes in exchange-driven aggregation not only has theoretical interest, but also can give a more realistic description for the kinetic evolution of natural or social systems. For example, the employee number of a company grows by attracting employees from other companies in the same industry or decreases due to the reverse process, which can be regarded as exchange-driven growth. Meanwhile, a company may hire novices to enlarge its production scale, provided that more and more agencies join in supplying its products, and the market share of its products thus increases continuously. This can be regarded as a catalyzed birth process with the help of product agencies. On the contrary, a company might also have to fire its employees as so to reduce its production scale if a new industry develops an alternative product of high quality to continuously invade and occupy the market, which can be modeled as a catalyzed death process caused by the existence of other companies in a different industry. Additionally, exchange-driven aggregate growth with competition between catalyzed birth and catalyzed death processes also occurs in many chemical systems. Motivated by these processes, with the growth or reduction in the sizes of the aggregates driven by other aggregates (especially aggregates of other species), we propose an exchange-driven aggregation model with competition between catalyzed birth and catalyzed death processes, which is believed to provide a fairly useful description of the dynamics of some physical (biological, chemical, and social) systems.

In our model, there are three distinct species of aggregates in the system, A, B, and C aggregates. Exchange-driven aggregation occurs between any two aggregates of the same species, and, meanwhile, an aggregate of species A gains a new monomer catalyzed by an aggregate of species B or loses a member catalyzed by species C. By discussing their evolution behaviors, we found that the aggregate size distribution of species A [the concentration of the aggregate A_k, $a_k(t)$] depends crucially on the competition between catalyzed birth and catalyzed death, as well as the exchange-driven growth. It obeys a conventional or generalized scaling law when the exchange-driven aggregation or the catalyzed birth dominates the process, while the aggregate size distribution evolves only according to modified scaling forms when the catalyzed death dominates the process.

The rest of this paper is organized as follows. In Sec. II, we introduce model of exchange-driven aggregation growth with competition between the catalyzed birth and catalyzed death processes, and describe the outline of the generalized Smoluchowski's rate equation approach to study the kinetic evolution behaviors of the aggregate size distributions. In Sec. III, we study the kinetics of the system in various cases with different dependences of catalyzed birth and catalyzed death rates of species A on the catalyst sizes. Finally, a brief summary is given in Sec. IV.

II. MODEL OF EXCHANGE-DRIVEN AGGREGATE GROWTH WITH COMPETITION BETWEEN CATALYZED BIRTH AND CATALYZED DEATH

In our model, the exchange-driven growth schemes of species A, B, and C are $A_k+A_j \xrightarrow{K_1(k,j)} A_{k-1}+A_{j+1}$, $B_k+B_j \xrightarrow{K_2(k,j)} B_{k-1}+B_{j+1}$, and $C_k+C_j \xrightarrow{K_3(k,j)} C_{k-1}+C_{j+1}$, respectively, with the migration rate kernels $K_1(k,j)$, $K_2(k,j)$, and $K_3(k,j)$. The reaction of monomer birth of species A catalyzed by species B is $A_k+B_j \xrightarrow{I(k,j)} A_{k+1}+B_j$, with the catalyzed birth rate kernel $I(k,j)$, and the reaction of monomer death of species A catalyzed by species C is $A_k+C_j \xrightarrow{J(k,j)} A_{k-1}+C_j$, with the catalyzed death rate kernel $J(k,j)$.

In general, the rate kernels of exchange and catalyzed birth or death are dependent on the reactant aggregate sizes. Here, for convenience of solving the rate equations, we focus on the typical symmetrical exchange kernels $K_1(k,j)=K_1kj$, $K_2(k,j)=K_2kj$, and $K_3(k,j)=K_3kj$, which are proportional to the sizes of aggregates migrating out and accepting one monomer (K_1, K_2, and K_3 are different proportionality constants). The rate kernel of species-B-catalyzed birth of species A is assumed to be $I(k,j)=Ikj^v$, and that of species-C-catalyzed death is assumed to be $J(k,j)=Jkj^v$, where I and J are also proportionality constants, and v is a parameter reflecting the dependence of the catalyzed birth and death rates on the catalyst aggregate size.

In this paper, we assume that the system has spatial homogeneity, so that the fluctuations in the densities of the reactants are ignored and the aggregates of each species are considered to be homogeneously distributed in space throughout the whole process. Thus, the theoretical approach to investigating the kinetics of the aggregation process can be based on the rate equations in the mean-field frame, which assumes that the reaction proceeds at a rate proportional to the reactant concentrations. We generalize the rate equation of the exchange- or migration-driven aggregation process [23,24] and write the corresponding rate equations for our system as follows:

$$\frac{da_k}{dt}=K_1(k+1)a_{k+1}\sum_{j=1}^{\infty}ja_j+K_1(k-1)a_{k-1}\sum_{j=1}^{\infty}ja_j - 2K_1ka_k\sum_{j=1}^{\infty}ja_j+I(k-1)a_{k-1}\sum_{j=1}^{\infty}j^vb_j-Ika_k\sum_{j=1}^{\infty}j^vb_j + J(k+1)a_{k+1}\sum_{j=1}^{\infty}j^vc_j-Jka_k\sum_{j=1}^{\infty}j^vc_j, \tag{1}$$

$$\frac{db_k}{dt}=K_2(k+1)b_{k+1}\sum_{j=1}^{\infty}jb_j+K_2(k-1)b_{k-1}\sum_{j=1}^{\infty}jb_j - 2K_2kb_k\sum_{j=1}^{\infty}jb_j, \tag{2}$$

$$\frac{dc_k}{dt}=K_3(k+1)c_{k+1}\sum_{j=1}^{\infty}jc_j+K_3(k-1)c_{k-1}\sum_{j=1}^{\infty}jc_j - 2K_3kc_k\sum_{j=1}^{\infty}jc_j, \tag{3}$$

where $b_k(t)$ and $c_k(t)$ are the concentrations of the aggregates B_k and C_k, respectively.

In Eq. (1), the first two terms account for the gain in $a_k(t)$ due to the migrations $A_{k+1}+A_l \xrightarrow{K(k+1,l)} A_k+A_{l+1}$ and $A_{k-1}+A_l \xrightarrow{K(l,k-1)} A_k+A_{l-1}$ $(l=1,2,\ldots)$, while their equiprobable reaction channels $A_{k+1}+A_l \xrightarrow{K(l,k+1)} A_{k+2}+A_{l-1}$ and $A_{k-1}+A_l \xrightarrow{K(k-1,l)} A_{k-2}+A_{l+1}$ give rise to gains in $a_{k+2}(t)$ and $a_{k-2}(t)$, which are accounted for in the rate equations da_{k+2}/dt and da_{k-2}/dt, respectively. The third term accounts for the loss in $a_k(t)$ due to the migration $A_k+A_l \xrightarrow{K(k,l)} A_{k-1}+A_{l+1}$ and its equiprobable process $A_k+A_l \xrightarrow{K(l,k)} A_{k+1}+A_{l-1}$. The fourth and fifth terms account, respectively, for the gain and loss in $a_k(t)$ due to species-B-catalyzed birth of species A, and the sixth and seventh terms account, respectively, for the gain and loss in $a_k(t)$ due to species-C-catalyzed death of species A.

For $a_1(t)$, the rate equation (1) has not the second and fourth terms of the equation for $a_k(t)$ $(k=2,3,\ldots)$, because there is no empty aggregate A_0, and similarly for $b_1(t)$ and $c_1(t)$. However, the rate equation of each species can be written in the above-mentioned corresponding uniform form for all $k\geqslant 1$ if the boundary condition $[a_0(t)=0,\ b_0(t)=0$, and $c_0(t)=0]$ is imposed.

The υth moment of the aggregate size distribution $a_k(t)$ of species A can be expressed as $M_\upsilon^A(t)=\Sigma_{j=1}^{\infty} j^\upsilon a_j(t)$. In particular, $M_0^A(t)=\Sigma_{j=1}^{\infty} a_j(t)$ and $M_1^A(t)=\Sigma_{j=1}^{\infty} j a_j(t)$ are the total number and the total mass of species A aggregates, respectively. That is the same for the species B and C.

Using the expression for the moments of the aggregate size distributions, the above equations (1)–(3) can be rewritten in the following forms:

$$\frac{da_k}{dt}=[K_1(k+1)a_{k+1}+K_1(k-1)a_{k-1}-2K_1ka_k]M_1^A \\ +[I(k-1)a_{k-1}-Ika_k]M_\upsilon^B+[J(k+1)a_{k+1}-Jka_k]M_\upsilon^C, \tag{4}$$

$$\frac{db_k}{dt}=[K_2(k+1)b_{k+1}+K_2(k-1)b_{k-1}-2K_2kb_k]M_1^B, \tag{5}$$

$$\frac{dc_k}{dt}=[K_3(k+1)c_{k+1}+K_3(k-1)c_{k-1}-2K_3kc_k]M_1^C. \tag{6}$$

Several methods have been developed to solve the rate equations in different aggregation processes. For the simple coagulation process $A_i+A_j \xrightarrow{K(i,j)} A_{i+j}$, explicit solutions were obtained by introducing some suitable generating functions [7–9]. In aggregation-annihilation processes, the rate equations were solved with the help of a special *Ansatz* $a_k(t)=A(t)[a(t)]^{k-1}$, and by the generating function method as well [15]. In exchange-driven aggregation processes, the rate equations were solved through making some scaling *Ansätze* directly [23,24] or with the help of the special *Ansatz* $a_k(t)=A(t)[a(t)]^{k-1}$ [25,26]. Recently, Lin *et al.* gave a thorough discussion of the applicability of this *Ansatz* [31]. Here we find that our current rate equations (4)–(6) can be solved with the help of the *Ansatz*

$$a_k(t)=A(t)[a(t)]^{k-1},\quad b_k(t)=B(t)[b(t)]^{k-1}, \\ c_k(t)=C(t)[c(t)]^{k-1}. \tag{7}$$

In a previous study of the mutually catalyzed birth of population and assets in exchange-driven growth [31], the species-B aggregates perform the exchange-driven aggregation process and the monomer birth process under the catalysis of species A; while in this exchange-driven aggregation model, the species-B and -C aggregates perform only the exchange-driven growth processes, namely, the species-A-catalyzed birth rates of both species-B and -C aggregates are equal to zero, so their evolution kinetic behaviors are simpler. Under the monodisperse initial condition $b_k(0)=B_0\delta_{k1}$ and $c_k(0)=C_0\delta_{k1}$ (B_0 and C_0 are the initial aggregate concentrations of species B and C, respectively), the accurate solution to the aggregate size distribution of species B (similarly for species C) is [31],

$$b_k(t)=B_0(1+K_2B_0t)^{-2}[1-(1+K_2B_0t)^{-1}]^{k-1}. \tag{8}$$

In the region of $t\gg 1$ and $k\gg 1$, Eq. (8) can be asymptotically rewritten as

$$b_k(t)\simeq K_2^{-2}B_0^{-1}t^{-2}\exp[-k(K_2B_0t)^{-1}]. \tag{9}$$

It satisfies the conventional scaling law in the long-time limit [15],

$$a_k(t)\simeq t^{-w}\Phi[k/S(t)],\quad S(t)\propto t^z, \tag{10}$$

with the scaling function $\Phi(x)=\exp(-x)$. $S_B(t)=K_2B_0t$ is the characteristic size of B aggregates, which plays a role analogous to the correlation length in critical phenomena. Moreover, the total number and total mass of B aggregates are derived as

$$M_0^B(t)=B_0(1+K_2B_0t)^{-1}\simeq K_2^{-1}t^{-1},\quad M_1^B(t)=B_0. \tag{11}$$

Thus, the total number decays linearly with time as t^{-1}, and the total mass is conserved.

We now focus on the evolution behavior of species A through solving the rate equation (4) under the monodisperse initial condition $a_k(0)=A_0\delta_{k1}$, where A_0 is the initial aggregate concentrations of species A. With the *Ansatz* (7), the first two moments of species A can be written as $M_0^A(t)=\Sigma_{j=1}^{\infty}a_j(t)=A(t)/[1-a(t)]$ and $M_1^A(t)=\Sigma_{j=1}^{\infty}ja_j(t)=A(t)/[1-a(t)]^2$, with the relation $M_0^A(t)=M_1^A(t)[1-a(t)]$. Substituting the *Ansatz* (7) into the rate equation (4), it can be transformed into the following differential equations:

$$\frac{1}{1-a}\frac{da}{dt}=K_1M_0^A+IM_\upsilon^B-JM_\upsilon^Ca, \tag{12}$$

$$\frac{1}{A}\frac{dA}{dt}=-2K_1M_0^A-IM_\upsilon^B+JM_\upsilon^C(2a-1). \tag{13}$$

The monodisperse initial condition is transformed correspondingly as follows:

$$a(0) = 0, \quad A(0) = A_0 \quad \text{at } t = 0. \tag{14}$$

First, we derive the relation between $A(t)$ and $a(t)$ from Eqs. (12) and (13) as

$$A(t) = A_0[1 - a(t)]^2 E_B(t) E_C^{-1}(t), \tag{15}$$

with the shorthand notations $E_B(t)=\exp[I\int_0^t M_v^B(t')dt']$ and $E_C(t)=\exp[J\int_0^t M_v^C(t')dt']$.

Substituting Eq. (15) into Eq. (12), it can be rewritten as

$$\frac{d(1-a)}{dt} = -(K_1 A_0 E_B E_C^{-1} + J M_v^C)(1-a)^2 + (J M_v^C - I M_v^B)(1-a). \tag{16}$$

This Bernoulli equation can be solved under the initial condition (14) to yield

$$a(t) = 1 - \frac{E_B^{-1}(t) E_C(t)}{1 + K_1 A_0 t + \int_0^t E_B^{-1}(t') dE_C(t')}. \tag{17}$$

Thus, the problem is reduced to deriving an explicit expression for $a(t)$ from Eq. (17). In the next section, we study the kinetics of the system in various cases with different catalyzed birth and death rate kernel parameters v, reflecting the dependence of the catalyzed birth and death rates on the catalyst aggregate size.

III. KINETICS OF EXCHANGE-DRIVEN AGGREGATION GROWTH WITH COMPETITION BETWEEN CATALYZED BIRTH AND CATALYZED DEATH

A. The case of $v=0$

We first study the relatively simple case of $v=0$, a case where the catalyzed birth and death rate kernels $I(k,j)=Ik$ and $J(k,j)=Jk$ are both independent of the catalyst aggregate size.

In this case, we can obtain an asymptotical explicit expression for $a(t)$ in the long-time limit. From Eq. (11), we have $M_0^B(t)=B_0(1+K_2B_0t)^{-1}$ and $M_0^C(t)=C_0(1+K_3C_0t)^{-1}$; then we obtain $E_B(t)=(1+K_2B_0t)^{I/K_2}$ and $E_C(t)=(1+K_3C_0t)^{J/K_3}$. Substituting these solutions into Eq. (17), we derive the analytic solution for $a(t)$ in the following several subcases.

In the subcase of $JK_3^{-1}-IK_2^{-1}<1$, from Eq. (17) we derive the following asymptotic solution in the long-time limit:

$$a(t) \simeq 1 - (C_1 K_1 A_0)^{-1} t^{-1-IK_2^{-1}+JK_3^{-1}}, \tag{18}$$

where $C_1=(K_2B_0)^{IK_2^{-1}}(K_3C_0)^{-JK_3^{-1}}$. We further derive the asymptotic solution of $A(t)$ from Eq. (15),

$$A(t) \simeq C_1^{-1} K_1^{-2} A_0^{-1} t^{-2-IK_2^{-1}+JK_3^{-1}}. \tag{19}$$

Thus, the aggregate size distribution of species A in the long-time limit is derived as

$$a_k(t) \simeq C_1^{-1} K_1^{-2} A_0^{-1} t^{-2-(IK_2^{-1}-JK_3^{-1})} \exp(-x), \quad x = k/S_A(t), \tag{20}$$

with the characteristic aggregate size $S_A(t)=C_1K_1A_0t^{1+(IK_2^{-1}-JK_3^{-1})}$. This shows that the aggregate size distribution $a_k(t)$ satisfies the conventional scaling form (10), with the scaling function $\Phi(x)=\exp(-x)$. The scaling exponents $w=2+IK_2^{-1}-JK_3^{-1}$ and $z=1+IK_2^{-1}-JK_3^{-1}$ are dependent on the rate kernels of the self-exchange of species B and C, as well as on the rate kernels of the catalyzed birth and death reactions, but they are independent of the self-exchange of species A itself. More precisely, the kinetic evolution behavior of species A is dominated by the value $IK_2^{-1}-JK_3^{-1}$, which reflects the competition between the species-B-catalyzed birth rate scaled by the rate of self-exchange of species B and the species-C-catalyzed death rate scaled by the rate of self-exchange of species C. This suggests that when both species B and C perform self-exchange processes, the effective species-B-catalyzed birth rate is $I_e=IK_2^{-1}$ and the effective species-C-catalyzed death rate is $J_e=JK_3^{-1}$.

Moreover, the total number of species A is derived as follows:

$$M_0^A(t) = A(t)[1-a(t)]^{-1} \simeq K_1^{-1} t^{-1}, \tag{21}$$

which decays with time as t^{-1}. The total mass of species A is

$$M_1^A(t) = A(t)[1-a(t)]^{-2} = A_0 E_B(t) E_C^{-1}(t) \simeq C_1 A_0 t^{IK_2^{-1}-JK_3^{-1}}. \tag{22}$$

Obviously, the evolution of the total mass $M_1^A(t)$ is crucially dominated by the competition between the effective catalyzed birth and the effective catalyzed death. For the $I_e>J_e$ case, the catalyzed birth dominates the process and $M_1^A(t)$ increases with time, while in the $I_e<J_e$ case, the catalyzed death dominates the process and $M_1^A(t)$ decreases with time. In the marginal case of $I_e=J_e$, the catalyzed birth and death processes play the same roles and counteract each other, and $M_1^A(t)$ is conserved.

In the subcase of $JK_3^{-1}-IK_2^{-1}=1$, we obtain the following asymptotic solution for $K_2B_0\neq K_3C_0$:

$$a(t) \simeq 1 - C_2 - C_3 t^{-1} \ln t, \tag{23}$$

where $C_2=(K_1A_0C_1+JK_3^{-1})^{-1}$ and $C_3=IJK_2^{-1}K_3^{-1}C_2^2(K_2^{-1}B_0^{-1}-K_3^{-1}C_0^{-1})$. For $K_2B_0=K_3C_0$, one can easily derive the exact solution of $a(t)$ from Eq. (16),

$$a(t) = \frac{(K_1A_0 + JC_0 - K_2B_0)t}{1 + (K_1A_0 + JC_0)t}, \tag{24}$$

which reduces to the following asymptotic solution at large times: $a(t)\simeq 1-C_2-(1-C_2)(K_1A_0+JC_0)^{-1}t^{-1}$. Substituting Eqs. (23) and (24) into Eq. (15), respectively, we can obtain the same expression for $A(t)$ as follows:

$$A(t) \simeq A_0 C_1 C_2^2 t^{-1}. \tag{25}$$

Thus we obtain the large-time asymptotic solution of the aggregate size distribution

$$a_k(t) \simeq \begin{cases} A_0C_1C_2^2(1-C_2)^k t^{-1} \exp[-kC_3(1-C_2)^{-1}t^{-1}\ln t] & \text{for } K_2B_0 \neq K_3C_0, \\ A_0C_1C_2^2(1-C_2)^k t^{-1} \exp[-k(K_1A_0+JC_0)^{-1}t^{-1}] & \text{for } K_2B_0 = K_3C_0. \end{cases} \tag{26}$$

Equation (26) shows that the usual scaling form (10) breaks down for this case and the aggregate size distribution satisfies the modified scaling form [15],

$$a_k(t) \simeq \lambda^k t^{-w}\Phi[k/S(t)], \quad S(t) \propto t^z. \tag{27}$$

It is well known that the usual scaling form has only one scale $S(t)$. Equation (27) shows that the modified scaling form comprises two scales, the growing scale $S(t)$ and the time-independent scale $S = \lim_{t\to\infty}\Sigma_k k^2 a_k(t)/\Sigma_k k a_k(t) = 1/(1-\lambda)$. The growing scale denotes the evolution of the aggregates driven by the exchange and catalyzed birth and death processes, while the time-independent scale dominates the kinetic behavior of the system in the long-time limit. For this subcase, the growing scale is $S(t) = (1-C_2)C_3^{-1}t(\ln t)^{-1}$ for $K_2B_0 \neq K_3C_0$ and $S(t) = (K_1A_0+JC_0)t$ for $K_2B_0 = K_3C_0$, while the time-independent scale is $S = C_1K_1A_0 + JK_3^{-1}$.

Moreover, we derive, respectively, the asymptotic solutions for the total number and the total mass of species A,

$$M_0^A(t) \simeq A_0C_1C_2t^{-1}, \quad M_1^A(t) \simeq A_0C_1t^{-1}. \tag{28}$$

So both the total number and the total mass of species A decay with time as t^{-1}, which is independent of the details of the rate kernels.

In the subcase of $JK_3^{-1} - IK_2^{-1} > 1$, the evolution behavior of the aggregate size distribution further depends on the relation between $(J-K_3)K_2^2B_0$ and $IK_3^2C_0$. When $(J-K_3)K_2^2B_0 \neq IK_3^2C_0$, from Eq. (17) we determine the following asymptotic solution for $a(t)$ in the long-time limit:

$$a(t) \simeq \begin{cases} 1 - C_4 + C_5 t^{-JK_3^{-1}+IK_2^{-1}+1} & \text{for } JK_3^{-1} - IK_2^{-1} < 2, \\ 1 - C_4 + C_6 t^{-1} & \text{for } JK_3^{-1} - IK_2^{-1} \geqslant 2, \end{cases} \tag{29}$$

where $C_4 = 1 - IK_3/JK_2$, $C_5 = K_1A_0C_1C_4^2$, $C_6 = C_5 + C_7$ for $JK_3^{-1} - IK_2^{-1} = 2$ or $C_6 = C_7(JK_3^{-1} - IK_2^{-1} - 1)^{-1}$ for $JK_3^{-1} - IK_2^{-1} > 2$, and $C_7 = JK_3^{-1}[(J-K_3)K_3^{-2}C_0^{-1} - IK_2^{-2}B_0^{-1}]C_4^2$. By making use of Eq. (29) we deduce the explicit solution for $A(t)$ from Eq. (15),

$$A(t) \simeq A_0C_1C_4^2t^{-1}. \tag{30}$$

Then we obtain the scaling solution of the aggregate size distribution,

$$a_k(t) \simeq \begin{cases} A_0C_1C_4^2(1-C_4)^k t^{-1} \exp[kC_5(1-C_4)^{-1}t^{-JK_3^{-1}+IK_2^{-1}+1}] & \text{for } JK_3^{-1} - IK_2^{-1} < 2, \\ A_0C_1C_4^2(1-C_4)^k t^{-1} \exp[kC_6(1-C_4)^{-1}t^{-1}] & \text{for } JK_3^{-1} - IK_2^{-1} \geqslant 2, \end{cases} \tag{31}$$

which takes the modified scaling form (27). For this subcase, the growing scale is $S(t) = (1-C_4)C_5^{-1}t^{JK_3^{-1}-IK_2^{-1}-1}$ for $1 < JK_3^{-1} - IK_2^{-1} < 2$ and $S(t) = (1-C_4)C_6^{-1}t$ for $JK_3^{-1} - IK_2^{-1} \geqslant 2$, while the time-independent scale is always $S = JK_2/(JK_2 - IK_3)$. When $(J-K_3)K_2^2B_0 = IK_3^2C_0$, the large-time asymptotic solution for $a(t)$ can be derived as follows:

$$a(t) \simeq \begin{cases} 1 - C_4 + C_5 t^{-JK_3^{-1}+IK_2^{-1}+1} & \text{for } JK_3^{-1} - IK_2^{-1} < 3, \\ 1 - C_4 + C_8 t^{-2} & \text{for } JK_3^{-1} - IK_2^{-1} \geqslant 3, \end{cases} \tag{32}$$

where $C_8 = C_5 + C_9$ for $JK_3^{-1} - IK_2^{-1} = 3$ or $C_8 = C_9(JK_3^{-1} - IK_2^{-1} - 2)^{-1}$ for $JK_3^{-1} - IK_2^{-1} > 3$ and $C_9 = JK_3^{-1}[I(I+K_2)K_2^{-4}B_0^{-2}/2 + (J-K_3)(J-2K_3)K_3^{-4}C_0^{-2}/2 - I(J-K_3)K_2^{-2}K_3^{-2}B_0^{-1}C_0^{-1}]C_4^2$. The solution for $A(t)$ is the same as Eq. (30). Thus, the scaling solution of the aggregate size distribution for $JK_3^{-1} - IK_2^{-1} < 3$ is the same as Eq. (29) for $JK_3^{-1} - IK_2^{-1} < 2$, while that for $JK_3^{-1} - IK_2^{-1} \geqslant 3$ can be written as

$$a_k(t) \simeq A_0C_1C_4^2(1-C_4)^k t^{-1} \exp[kC_7(1-C_4)^{-1}t^{-2}], \tag{33}$$

with the growing scale $S(t) = C_7^{-1}(1-C_4)t^2$ and the time-independent scale $S = JK_2/(JK_2 - IK_3)$. Moreover, the asymptotic solutions of the total number and the total mass of species A is obtained as follows:

$$M_0^A(t) \simeq A_0C_1C_4t^{-(JK_3^{-1}-IK_2^{-1})}, \quad M_1^A(t) \simeq A_0C_1t^{-(JK_3^{-1}-IK_2^{-1})}. \tag{34}$$

The results imply that both the total number and the total mass of species A decrease as $t^{-(JK_3^{-1}-IK_2^{-1})}$ in the long-time limit. So species A will also die out finally in this subcase.

B. The case of $v = 1$

We now discuss the case of $v = 1$, where the catalyzed birth and death rate kernels $I(k,j) = Ikj$ and $J(k,j) = Jkj$ are both proportional to the catalyst's size.

In this case, the total masses of species B and C are both conserved, $M_1^B(t)=B_0$ and $M_1^C(t)=C_0$. The expressions for $E_B(t)$ and $E_C(t)$ can be directly derived as $E_B(t)=\exp(IB_0t)$ and $E_C(t)=\exp(JC_0t)$. The solution for $a(t)$ can be exactly derived from Eq. (17) as

$$a(t)=\begin{cases}1-\dfrac{(IB_0-JC_0)e^{-(IB_0-JC_0)t}}{IB_0-JC_0+(IB_0-JC_0)K_1A_0t-JC_0e^{-(IB_0-JC_0)t}} & \text{for } IB_0\neq JC_0,\\ 1-(K_1A_0+JC_0)^{-1}t^{-1} & \text{for } IB_0=JC_0.\end{cases} \tag{35}$$

In the long-time limit, it becomes

$$a(t)\simeq\begin{cases}1-(K_1A_0)^{-1}t^{-1}e^{-(IB_0-JC_0)t} & \text{for } IB_0>JC_0,\\ 1-(K_1A_0+JC_0)^{-1}t^{-1} & \text{for } IB_0=JC_0,\\ IB_0(JC_0)^{-1}+K_1A_0[1-IB_0(JC_0)^{-1}]^2te^{-(JC_0-IB_0)t} & \text{for } IB_0<JC_0.\end{cases} \tag{36}$$

We further derive the asymptotic solution for $A(t)$ from Eq. (15) as

$$A(t)\simeq\begin{cases}K_1^{-2}A_0^{-1}t^{-2}e^{-(IB_0-JC_0)t} & \text{for } IB_0>JC_0,\\ A_0(K_1A_0+JC_0)^{-2}t^{-2} & \text{for } IB_0=JC_0,\\ A_0[1-IB_0(JC_0)^{-1}]^2e^{-(JC_0-IB_0)t} & \text{for } IB_0<JC_0.\end{cases} \tag{37}$$

(i) When $IB_0>JC_0$, the aggregate size distribution of species A is obtained in the long-time limit satisfying a generalized scaling form as

$$a_k(t)\simeq K_1^{-2}A_0^{-1}t^{-2}e^{-(IB_0-JC_0)t}\exp(-x),\quad x=k/S_A(t), \tag{38}$$

with the characteristic size of aggregates $S_A(t)=K_1A_0te^{(IB_0-JC_0)t}$. This generalized scaling expression of the aggregate size distribution can further be written in a general form as follows [31]:

$$a_k(t)\simeq K_1^{-1}t^{-1}S_A^{-1}(t)\Phi(x),\quad x=k/S_A(t). \tag{39}$$

The total number and the total mass of species A can be derived as

$$M_0^A(t)\simeq K_1^{-1}t^{-1},\quad M_1^A(t)\simeq A_0e^{(IB_0-JC_0)t}. \tag{40}$$

The results imply that the total number of species A still decays as t^{-1}, while the total mass grows exponentially with time. This means that catalyzed birth dominates the process.

(ii) For the $IB_0=JC_0$ case, the aggregate size distribution of species A satisfies the conventional scaling form in the long-time limit,

$$a_k(t)\simeq A_0(K_1A_0+JC_0)^{-2}t^{-2}\Phi(x),\quad x=k/S_A(t), \tag{41}$$

with the characteristic size $S_A(t)=(K_1A_0+JC_0)t$ and the scaling exponents $w=2$ and $z=1$.

Moreover, the total number and the total mass of A species can be obtained as

$$M_0^A(t)\simeq A_0(K_1A_0+JC_0)^{-1}t^{-1},\quad M_1^A(t)=A_0. \tag{42}$$

The results imply that the total number of species A still decays with time as t^{-1}, while the total number is conserved all along. It is just the result of the counteraction of catalyzed birth and catalyzed death.

(iii) When $IB_0<JC_0$, the aggregate size distribution of species A is obtained in the long-time limit as

$$a_k(t)\simeq A_0[1-IB_0(JC_0)^{-1}]^2(IB_0/JC_0)^ke^{-(JC_0-IB_0)t}\times\exp[k/R(t)], \tag{43}$$

with $R(t)=IB_0JC_0(JC_0-IB_0)^{-2}(K_1A_0)^{-1}t^{-1}e^{(JC_0-IB_0)t}$. It shows that, in this case, the aggregate size distribution $a_k(t)$ takes the modified scaling form (27). Moreover, the total number and the total mass of species A can be obtained as

$$M_0^A(t)\simeq A_0(JC_0-IB_0)(JC_0)^{-1}e^{-(JC_0-IB_0)t},$$

$$M_1^A(t)=A_0e^{-(JC_0-IB_0)t}. \tag{44}$$

So both the total number $M_0^A(t)$ and the total mass $M_1^A(t)$ decrease exponentially. That is, the effect of catalyzed death is greater than that of catalyzed birth, and species A will die out in the end.

Contrary to the former $v=0$ case, the self-exchange processes of species B and C play no role in the kinetic behavior of species A in this case. Meanwhile, the initial concentrations of species B and C, B_0 and C_0, play important roles in the evolution of the system. The kinetic behavior of species A is dominated by the competition between the effective species-B-catalyzed birth $I_e=IB_0$ and the effective species-C-catalyzed death $J_e=JC_0$.

C. The general v case

In the general v case, we can study the asymptotic kinetic behavior of species A in the long-time limit.

In the long-time limit, from Eq. (9) the vth moment of the distribution $b_k(t)$ can be determined as

$$M_v^B(t) = \sum_{j=1}^{\infty} j^v b_j \simeq \begin{cases} \Gamma(1+v)B_0(1+K_2B_0t)^{v-1} & \text{for } v > -1, \\ (K_2t)^{-1}(1+K_2B_0t)^{-1}\ln(1+K_2B_0t) & \text{for } v = -1, \\ B_0(1+K_2B_0t)^{-2}\zeta(-v) & \text{for } v < -1, \end{cases} \tag{45}$$

where $\zeta(n)=\Sigma_{l=1}^{\infty} l^{-n}$ $(n>1)$ is the Riemann zeta function. The vth moment of the distribution $c_k(t)$ is

$$M_v^C(t) = \sum_{j=1}^{\infty} j^v c_j \simeq \begin{cases} \Gamma(1+v)C_0(1+K_3C_0t)^{v-1} & \text{for } v > -1, \\ (K_3t)^{-1}(1+K_3C_0t)^{-1}\ln(1+K_3C_0t) & \text{for } v = -1, \\ C_0(1+K_3C_0t)^{-2}\zeta(-v) & \text{for } v < -1. \end{cases} \tag{46}$$

(i) In the case of $v>0$, from Eqs. (45) and (46) we derive

$$E_B(t) \simeq \exp(C_{11}t^v), \tag{47}$$

$$E_C(t) \simeq \exp(C_{12}t^v), \tag{48}$$

where $C_{11}=I\Gamma(v)B_0^vK_2^{v-1}$ and $C_{12}=J\Gamma(v)C_0^vK_3^{v-1}$.

Using these expressions, we obtain $a(t)$ from Eq. (17),

$$a(t) \simeq \begin{cases} 1-(K_1A_0)^{-1}t^{-1}e^{-(C_{11}-C_{12})t^v} & \text{for } C_{11} > C_{12}, \\ 1-(K_1A_0t+C_{12}t^v)^{-1} & \text{for } C_{11} = C_{12}, \\ C_{11}C_{12}^{-1}+(C_{12}C_{11}^{-1}-1)^2K_1A_0te^{-(C_{12}-C_{11})t^v} & \text{for } C_{11} < C_{12}. \end{cases} \tag{49}$$

We further derive the asymptotic solution of $A(t)$ from Eq. (15),

$$A(t) \simeq \begin{cases} K_1^{-2}A_0^{-1}t^{-2}e^{-(C_{11}-C_{12})t^v} & \text{for } C_{11} > C_{12}, \\ A_0(K_1A_0t+C_{12}t^v)^{-2} & \text{for } C_{11} = C_{12}, \\ A_0(1-C_{11}C_{12}^{-1})^2e^{-(C_{12}-C_{11})t^v} & \text{for } C_{11} < C_{12}. \end{cases} \tag{50}$$

It follows from Eqs. (49) and (50) that the kinetic behavior of the aggregate size distribution $a_k(t)$ depends crucially on the relationship of C_{11} and C_{12}, which reflects the competition between catalyzed birth and catalyzed death. So, in this general $v>0$ case, Eqs. (49) and (50) suggest that the effective rates of species-B-catalyzed birth and species-C-catalyzed death are $I_e=C_{11}=I\Gamma(v)B_0^vK_2^{v-1}$ and $J_e=C_{12}=J\Gamma(v)C_0^vK_3^{v-1}$, respectively.

For the $C_{11}>C_{12}$ case, in which the catalyzed birth dominates the process, the aggregate size distribution of species A can be described in the generalized scaling form

$$a_k(t) = K_1^{-1}t^{-1}S_A^{-1}(t)\exp(-x), \quad x=k/S_A(t), \tag{51}$$

with the characteristic size $S_A(t)=K_1A_0te^{(C_{11}-C_{12})t^v}$. The total number and the total mass of species A are obtained as

$$M_0^A(t) \simeq K_1^{-1}t^{-1}, \quad M_1^A(t) \simeq A_0\exp[(C_{11}-C_{12})t]. \tag{52}$$

The results imply that the total number of species A still decreases as t^{-1}, while the total mass grows exponentially with time.

For the $C_{11}=C_{12}$ case, in which the effects of catalyzed birth and catalyzed death counteract each other, the evolution behavior of the aggregate size distribution $a_k(t)$ is dependent on the parameter v.

If $0<v<1$, the aggregate size distribution of species A can be expressed in the conventional scaling form

$$a_k(t) \simeq K_1^{-2}A_0^{-1}t^{-2}\exp(-x), \quad x=k/S_A(t), \tag{53}$$

with the characteristic size $S_A(t)=K_1A_0t$. The scaling exponents are $w=2$ and $z=1$, which are independent of the value of the parameter v. The total number and the total mass of species A are derived as

$$M_0^A(t) \simeq K_1^{-1}t^{-1}, \quad M_1^A(t) = A_0. \tag{54}$$

The results indicate that the total number of species A decreases with time as t^{-1}, while the total mass is conserved.

On the other hand, if $v>1$, the aggregate size distribution of species A also satisfies the conventional scaling form

$$a_k(t) \simeq A_0C_{12}^{-2}t^{-2v}\exp(-x), \quad x=k/S_A(t), \tag{55}$$

with the characteristic size $S_A(t)=C_{12}t^v$, but the scaling exponents $w=2v$, $z=v$ are dependent on the value of the parameter v. The total number and the total mass of species A are derived as

$$M_0^A(t) \simeq A_0C_{12}^{-1}t^{-v}, \quad M_1^A(t) = A_0. \tag{56}$$

The results indicate that the total number of species A decreases with time as t^{-v}, while the total mass is also conserved.

For the $C_{11}<C_{12}$ case, which corresponds to the case where the catalyzed death dominates the process, the aggregate size distribution of species A can be expressed as

$$a_k(t) \simeq A_0(1 - C_{11}C_{12}^{-1})^2(C_{11}/C_{12})^k e^{-(C_{12}-C_{11})t^v} \times \exp[k/R(t)],$$

$$R(t) = C_{13}(K_1A_0)^{-1}t^{-1}e^{(C_{12}-C_{11})t^v}, \tag{57}$$

where $C_{13}=C_{11}C_{12}^{-1}(1-C_{12}C_{11}^{-1})^{-2}$. It is obvious that, in this case, the aggregate size distribution $a_k(t)$ obeys another modified scaling form,

$$a_k(t) \simeq \lambda^k t^{-w}[R(t)]^{-1}\Psi[k/R(t)], \tag{58}$$

with the scaling function $\Psi(x)=\exp(x)$. Moreover, both the total number $M_0^A(t)$ and the total mass $M_1^A(t)$ of species A decrease stretched-exponentially as $\exp[-(C_{12}-C_{11})t^v]$. So, in this case, species A will also die out finally.

(ii) In the case of $v<0$, from Eqs. (17), (45), and (46) one can easily obtain the asymptotical solution of $a(t)$ in the long-time limit as

$$a(t) \simeq 1 - C_{14}(K_1A_0t)^{-1}, \tag{59}$$

where $C_{14}=\exp\{\int_0^\infty[JM_v^C(t)-IM_v^B(t)]dt\}$. We further derive the asymptotic solution for $A(t)$ from Eq. (15) as

$$A(t) \simeq C_{14}K_1^{-2}A_0^{-1}t^{-2}. \tag{60}$$

The aggregate size distribution of species A is obtained and satisfies the conventional scaling form

$$a_k(t) \simeq C_{14}K_1^{-2}A_0^{-1}t^{-2}\exp(-x), \quad x = k/S_A(t), \tag{61}$$

with the characteristic size $S_A(t)=C_{14}^{-1}K_1A_0t$. The scaling exponents are $w=2$, $z=1$, which are independent of the value of the parameter v. The total number $M_0^A(t)$ and the total mass $M_1^A(t)$ of species A can be obtained as follows:

$$M_0^A(t) \simeq K_1^{-1}t^{-1}, \tag{62}$$

$$M_1^A(t) \simeq A_0 \exp\left(\int_0^\infty [IM_v^B(t) - JM_v^C(t)]dt\right).$$

From Eq. (62) we find that the total number of species A still decreases with time as t^{-1}, and the evolution of the total mass $M_1^A(t)$ is dominated by the competition between the catalyzed birth and catalyzed death reactions. When $\int_0^\infty[IM_v^B(t)-JM_v^C(t)]dt>0$, the total mass $M_1^A(t)$ grows with time, which reveals that the influence of catalyzed birth is greater than that of catalyzed death. When $\int_0^\infty[IM_v^B(t)-JM_v^C(t)]dt<0$, $M_1^A(t)$ decreases with time, which shows that the influence of catalyzed death is greater than that of catalyzed birth; and when $\int_0^\infty[IM_v^B(t)-JM_v^C(t)]dt=0$, $M_1^A(t)$ is conserved, which corresponds to the fact that the effects of catalyzed birth and catalyzed death counteract each other.

TABLE I. Form of the aggregate size distribution of species A.

v	Scaling behavior of $a_k(t)$	
$v<0$	Conventional scaling form	
$v=0$	Conventional scaling form	$(JK_3^{-1}-IK_2^{-1}<1)$
	Modified scaling behavior	$(JK_3^{-1}-IK_2^{-1}\geqslant-1)$
$v>0$	Generalized scaling form	$(IB_0^vK_2^{v-1}>JC_0^vK_3^{v-1})$
	Conventional scaling from	$(IB_0^vK_2^{v-1}=JC_0^vK_3^{v-1})$
	Modified scaling behavior	$(IB_0^vK_2^{v-1}<JC_0^vK_3^{v-1})$

IV. SUMMARY

In summary, we have proposed an exchange-driven growth model of a three-species system, in which exchange-driven aggregation occurs between any two clusters of the same species and, meanwhile, monomer birth and monomer death of species A occur due to catalysis by species B and C, respectively. Based on the mean-field rate equations, we analyzed the kinetic scaling behavior of the aggregate size distributions in the systems with the catalysis birth (death) rate kernels $I(i,j)=Iij^v$ $[J(i,j)=Jij^v]$. The kinetics of the systems are found to depend crucially on the value of v, as illustrated in Table I.

From Table I, we can draw the following conclusions. In the $v<0$ case, where the effects of catalyzed birth and catalyzed death are rather weak and the exchange-driven aggregation dominates the process, the aggregate size distribution $a_k(t)$ obeys the conventional scaling law. As the value of v increases, the effects of catalyzed birth and catalyzed death get larger, and when catalyzed birth dominates the process ($IK_2^{-1}-JK_3^{-1}>-1$ in the $v=0$ case and $IB_0^vK_2^{v-1}>JC_0^vK_3^{v-1}$ in the $v>0$ case), $a_k(t)$ follows the conventional or generalized scaling law. On the other hand, when catalyzed death dominates the process ($IK_2^{-1}-JK_3^{-1}\leqslant-1$ in the $v=0$ case and $IB_0^vK_2^{v-1}<JC_0^vK_3^{v-1}$ in the $v>0$ case), $a_k(t)$ scales according to some modified scaling laws, that is, the catalyzed death mechanism causes a breakdown of the conventional or generalized scaling behavior. In the marginal case when the effects of catalyzed birth and catalyzed death counteract each other in the $v>0$ case ($IB_0^vK_2^{v-1}=JC_0^vK_3^{v-1}$), exchange-driven aggregation dominates the process and $a_k(t)$ follows the conventional law, but it does not belong to the same universality class as the pure exchange-driven aggregation process.

Indeed, the competition between the catalyzed birth and death processes in exchange-driven aggregation gives rise to surprisingly rich kinetic behavior.

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